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Preface

This book is a personal attempt to convey my excitement about a circle of ideas that has captivated my imagination since my schooldays. At the core of these ideas are a series of ground-shaking discoveries and debates that rocked mathematics to its foundations during the late 19th and early 20th centuries. But the tremors from these upheavals are still being felt today, the most fundamental dilemmas are still unresolved, and for many, the sense of mystery at the heart of mathematics remains undiminished. The story of these ideas, with their close intertwining of mathematics, logic and philosophy and their relentless probing into some of the deepest questions we can ask, is for me one of the most fascinating chapters in the history of human thought.

I can pinpoint very precisely the moment when my interest in these ideas was awakened. I was 11, and my maths teacher, Mr. Orton, noting my enthusiasm for the subject, had taken me up to the senior school library and given me the chance to borrow any books that might interest me. My eye was caught by a small, unassuming red volume called *Gödel's Proof*, by Nagel and Newman.

"What's that?" I asked.

"Gödel," my teacher replied, "was a mathematician who showed that there are some mathematical statements that can be neither proved nor disproved."

Actually, as we'll see in Chapter 7 of this book, one has to be a bit more careful than this in order to say *exactly* what Gödel proved. But as a one-line summary for an inquisitive 11-year-old, it was appropriate, and it did the trick. I was electrified. Here was a mathematician who had proved something, not about numbers, or triangles, or graphs, or probabilities, but about *mathematics itself*. Furthermore, what he had proved sounded utterly astonishing and completely contrary to my expectations. Here, surely, was something profound, something mind-stretching, something I simply *had* to know about.

I took the book home and started to read it. I certainly didn't understand all of it at the time, but it gave me my first contact with the world of Frege, Russell, Hilbert and Gödel himself, and a seed had been planted.

A second brush with these ideas came about five years later, when I started to worry deeply over the concept of *infinity*. I had read and heard about certain kinds of infinite sets which everyone else seemed to think 'existed', but I myself wasn't so sure. I tried to formulate my ideas and doubts in a coherent way, but was disturbed to find that my beliefs seemed to contradict one another. It was only later that I learned that others had pursued these questions much further than I had, and that they formed part of the same overall story as Gödel's epoch-making result.

Purpose of this book

I have felt for a long time that this whole circle of ideas, touching as it does on deep questions concerning the nature of mathematics and of human rationality itself, was far too interesting and important to be confined to a small circle of specialist mathematicians and logicians. Indeed, many popular books (from Nagel and Newman's onwards) have done much to convey some of these ideas, especially Gödel's theorems, to a wider readership. But there still remains, as I see it, a much bigger intellectual story of which these ideas form a part — a story with roots in antiquity, but which continues to unfold in our present time — a coherent and gripping story deserving of a wide audience, but one which no previous popular account has sought to convey with the fullness it seems to demand.

About twenty years ago, I first conceived the idea of trying to portray this story in a visualizable form through the medium of a fantasy novel — much as George Gamow's classic *Mr. Tompkins* books had done for the ideas of relativity theory and quantum mechanics. Almost immediately, a riot of images started to invade my mind: an enormous castle, replete with magic rooms, precious treasures, grinding machinery, vicious serpents, transcendental staircases, and so on. After several years of this mental turmoil, I decided there was nothing for it but to sit down and try to write the book.

What is the book really about? In a nutshell, its main purpose is to introduce the reader to a big picture: namely, to a whole spectrum of possible ways of thinking about mathematics, its meaning, its extent, and its very nature. What, ultimately, *is* mathematics? Is it a window onto some realm of absolute truth? A free creation of our own minds? A natural outgrowth of logic or language? Or in the end just an elaborate game we play with symbols? Why is it that mathematical knowledge appears more certain than other kinds of knowledge? Is the whole of mathematics really 'certain' beyond a shadow of doubt, or are some parts more questionable than others? How is it that our minds can grasp mathematical truths, for example about infinite sets? Could a computer perceive such truths in the way a human can? How much of mathematics has any real meaning in any case?

These are deep questions, and as we shall see, the range of answers that have been suggested is enormous, often resulting in intense and heated debates between opposing camps. Interestingly, a key voice in these debates is that of mathematics itself, which (as Gödel found) has some remarkable things to tell us about its own nature. The purpose of this book, then, is to offer a mental map of this whole labyrinth of ideas, introducing you to the relevant mathematical concepts and to the philosophical questions and debates they give rise to.

Needless to say, all of this will take some time to develop. In order to convey the big picture I've just outlined, I shall take you on a journey — a journey of discovery, in which various parts of the picture will be revealed bit by bit. Some elements of this picture are already quite widely known, while others have never previously been presented at a popular level. We shall start with some classic ideas from number theory to set the scene, then move on to explore a number of key concepts from mathematical logic. Later, as the story develops, the more philosophical questions will come increasingly to the fore, although no previous knowledge of philosophy will be assumed. Along the way, I can promise you an intellectual adventure that will be intriguing, varied, instructive, challenging, tortuous, bizarre, and at times downright infuriating. Bertrand Russell, in his *History of Western Philosophy*, wrote that the exercise of learning to view the world through different philosophical spectacles was 'an imaginative delight and an antidote to dogmatism', and I hope that you will experience some of this enjoyment for yourself in the course of this book. In order to explore this spectrum of ideas, I frequently have my main characters voicing conflicting views — naturally, you are welcome to agree sometimes with one, sometimes with the other, or perhaps sometimes with neither. The one thing I certainly *can't* promise is any set of final, definitive answers.

A big picture by its very nature calls for a certain breadth of vision, and it's only fair to declare at the outset that I've been unashamedly ambitious in the ground I've set out to cover. Not all the ideas are easy, and they may take some grappling with. I should also acknowledge that some of the topics covered fall outside my own area of professional expertise, and I hope the experts will forgive me for the points where this shows. However, I've done my best to convey the broad sweep of the ideas as vividly and accessibly as I can, assuming no more than some school level maths and a fair dose of intellectual curiosity. The main protagonists in my story are two children; and while this isn't a 'children's book' as such, I've tried to create something that will be enjoyed by interested children and adults alike.

How to read this book

In many ways this book is something of an experiment. In our story, mathematical objects are portrayed as though they were physical objects that can be seen, touched and moved around. My hope is that this metaphorical approach will offer an accessible way in to some regions of thought that might otherwise seem impenetrable and obscure. Of course, the metaphors won't be perfect: most of them break down pretty quickly once you start to scrutinize them, and even to get them to work approximately I've had to resort to Hogwarts-style magic at almost every turn. There are also plenty of ingredients that don't have any metaphorical significance but are simply there for the sake of the story. Consequently, there is a danger that this approach could be misleading if the imagery is pressed too far. Nonetheless, it seems to me that the potential gains far outweigh the risks, and I hope that the story will serve as an entertaining introduction to the circle of ideas that make up mathematical logic.

But there is also something else I'm hoping to achieve by this approach. To some extent, I'm trying to let you in on the way mathematicians themselves think. There is a widespread perception that maths is all about juggling fearsome complexes of symbols — which is perhaps natural, because that's the way in which mathematics is typically communicated. But when mathematicians think about their subject, they will often draw on a much broader range of thought processes. The nature of these would seem to vary widely from one mathematician to another; but speaking personally, I frequently find myself thinking in pictures and images, or in loose physical analogies — perhaps even in emotions and bodily sensations. Translating all this into obscure symbols often comes later, and can even seem like a necessary evil, because for many mathematicians, it's the informal intuitions that get closer to the heart of what they're doing. It is a pity that many people, if their mathematical education has focused too narrowly on the symbol-juggling, never come into contact with the delightful, playful world of concepts to which the symbols refer. It's this imaginative, creative aspect of mathematics that I'm hoping to get across in this book.

Several ways of reading the book are possible. One way would be simply to read the story by itself, steadfastly ignoring all other distractions. This would be enough to give you an overall impression of the territory and a guided tour of some of its major landmarks. However, the story alone shouldn't be expected to deliver a thorough, in-depth understanding of the ideas: there is too much that could be misconstrued without further guidance. For readers wanting just a little more intellectual orientation, I've provided brief commentaries on each chapter that touch on the historical background and give a bit of a steer on what to take from the story (see Appendix A). For readers in quest of a more complete understanding, I am also developing extended versions of these commentaries which will be made freely available from the book's website:[1]

`www.castlesbook.org`

1 At the time of going to press, only a selection of these extended commentaries have been published on the website. Work on the remaining commentaries is ongoing.

The website also offers a number of other supplements providing more detail on specific topics. All in all, my aim has been to create a body of work which, like the Castle it portrays, offers something satisfyingly large and complex that can be explored with enjoyment on many levels.

But whatever approach you take, my most important advice would be this. Please, *please* don't worry if you don't understand everything the first time round! Many of the ideas simply *are* difficult and counterintuitive and mind-bending, however they're presented. So please don't feel that you need to grasp each idea completely before continuing with the next bit: it's absolutely fine just to read on and let the maths or the philosophy wash over you (though I realize this may be easier said than done if you're someone who likes to understand things properly). Indeed, I would be very happy to think this was the kind of book one could return to many times, understanding a little more with each re-reading.

You will find that the level of detail in which ideas are presented varies considerably from one part of the book to another. Sometimes we'll content ourselves with an aerial view of a whole tract of territory, arriving rapidly at conclusions that may have taken real-world mathematicians years to reach. At other times we shall proceed much more slowly and carefully, zooming in on the individual steps of a particular argument. In order to highlight likely sources of confusion, I sometimes have my characters making mistakes or missing things, but arriving at a correct understanding shortly afterwards. I've also planted little hints within the story to indicate the level of understanding you might reasonably expect to gain at various points. If our heroines seem to have just a rough idea of what's going on, then you too are allowed to have just a rough idea of it; and if they seem completely befuddled by some new concept, you're allowed to be completely befuddled too. The physicist Niels Bohr once remarked that anyone who hasn't been shocked by quantum mechanics hasn't understood it yet, and the same might well be said of many of the ideas of mathematical logic. Nevertheless, I do hope that as the book progresses, the strange and paradoxical will start to make its own kind of sense, and you will begin to enjoy that sense of an expanded horizon that comes with acquiring new ways of thinking.

Mathematical formulae

Whilst on the subject of accessibility, a few words about the mathematical formulae that you will find in some parts of the book.

Ever since Stephen Hawking boldly defied well-meaning advice by including Einstein's famous equation $E = mc^2$ in his bestseller *A Brief History of Time*, it has been traditional for agents and publishers to wage war against the use of formulae and equations in popular science books, and for the authors of such books to think they know better. In this book, the fantasy imagery has enabled me to dispense with mathematical notation in many places, but by no means everywhere — so a few words of explanation are in order.

First, I'd like to suggest that in the context of a book on mathematical logic, I have an unusually good excuse for including at least *some* symbolic notation in the text. This is because one of the cornerstones of the subject is precisely the idea that mathematical statements can be expressed in formal, symbolic languages, where they can be manipulated according to purely mechanical rules. In other words, in much of mathematical logic, symbolic formulae are themselves the *objects of study* — not just the means of studying something else, as they might be in other sciences or even other areas of mathematics. From this point of view, a book on mathematical logic that didn't include examples of logical formulae would be like a book on genetics that didn't include a diagram of a DNA molecule, or a book on geophysics that didn't show a cross-section of the Earth. My suggestion would be that when such formulae and the rules for manipulating them first appear in Chapters 4 and 6, you take the time to get your head round at least a sample of them — but after that, feel free to give such formulae as much or as little attention as you like.

Secondly, you will also find a smaller number of formulae that don't fall into this 'objects of study' category. Some of these, like the arithmetic examples in the early chapters, involve sufficiently basic mathematics that I don't think anyone will be deterred by them. But in a few other instances, I've allowed myself the use of mathematical notation to summarize something that's explained in words in the surrounding text. I can promise you that all such uses of notation will be short and sweet, of the $E = mc^2$ variety. I hope that when we come to such formulae, you will find that they help rather than hinder our thinking.

Matters of history

Next, a few comments on the historical content of the book.

Firstly, although the storyline introduces the ideas in a certain order, I should stress that this doesn't reflect the order in which the ideas were originally discovered. Rather, it's simply a convenient way in which these ideas can be connected up into a coherent narrative — a way partly dictated by the fantasy story format, and partly motivated by the philosophical questions I'm wanting to focus on. Not that the history of the ideas is unimportant or uninteresting — indeed, historical perspectives form a major strand of the commentaries that accompany the story. But in order to reveal the *logical* structure of the ideas, it seems best to take advantage of hindsight to fit them together in the simplest way possible. As a counterbalance to this, in Appendix B I've provided a historical timeline to show how these ingredients fit together chronologically.

A second, related caveat concerns the role played by the various mathematicians and philosophers who appear in the story. Each of these is introduced as a kind of 'figurehead' for a certain circle of ideas with which he is associated. (Yes, I'm afraid there *is* a severe gender imbalance in the history of the subject — something which I hope that my two heroines may do just a little to mitigate.) This usually means that the historical figure in question can be credited with the *main* idea under discussion, but not necessarily that he was responsible for absolutely everything I've put into his mouth. Although my characters will often use the actual words of their historical counterparts, at other times they will draw on ideas and insights that were reached only later. In the commentaries I will be a little more explicit about the history — but the story by itself should not be taken as a reliable guide to who said what and when.

From today's standpoint, the historic intellectual achievements of Cantor, Russell, Hilbert, Brouwer, Gödel, Turing and the rest can all too easily seem like the legendary doings of a mythical race of giants. In an attempt to give something of a human face to these figures, I've made use in the story of certain known details of their appearance, habits and personality, picked up from various sources — although I've done this without too much consistency, sometimes exaggerating their attributes to the point of caricature, sometimes supplementing them with characteristics of my own imagining where reliable details have been in short supply. So once again, these portrayals should not be taken as definitive biography. I would also hope that, despite occasional touches of wilful mockery, my overriding respect for all of these figures will shine through, and that my whimsical portraits will be taken in the playful spirit in which they are intended.

Influences and acknowledgements

The influence of many other books will be apparent in this one. I've already mentioned George Gamow's *Mr. Tompkins* as one of my chief inspirations. In addition, I have been stimulated by Raymond Smullyan's popular books on logic and by Rudy Rucker's *Infinity and the Mind*, while the indebtedness to Douglas Hofstadter's *Gödel, Escher, Bach* is (almost) too obvious to mention. Jostein Gaarder's *Sophie's World* encouraged me to think that the format I had in mind for the book was workable.

As regards the actual contents of the book, the work of the late Solomon Feferman (much of it collected in the volume *In the Light of Logic*) has had an enormous shaping effect on my vision of the subject area; I hope that he would have appreciated what I am trying to achieve here. Regarding particular parts of the book, Marcus du Sautoy's *The Music of the Primes* has had a significant influence on Chapters 1 to 3; Martin Davis's *Engines of Logic* on Chapters 4 and 5; David Bostock's *Philosophy of Mathematics* on Chapter 6; and Torkel Franzén's *Gödel's Theorem* on Chapter 7. Roger Penrose's books *The Emperor's New Mind* and *Shadows of the Mind* provided a major stimulus for some of the content in later chapters. Other sources from which I have drawn are listed in the bibliography. But it would be impossible for me to enumerate all the books, papers, talks and conversations that have fed my understanding over the years and whose influence may be discernible to those in the know. I can only record in general terms my gratitude for everything that has gone into the mix.

My interest in logic, already well established in my schooldays, was further nurtured at Cambridge University by my mentor Martin Hyland, as well as by many late nights spent fighting through Jon Barwise's *Handbook of Mathematical Logic* and train journeys in the company of Paul Cohen's *Set Theory and the Continuum Hypothesis*. For most of the time since then, I have benefited from the wonderful learning environment offered by the Laboratory for Foundations of Computer Science (LFCS) at the University of Edinburgh. I am also especially grateful for everything I have learned from collaborating with Dag Normann of the University of Oslo, my co-author on another book project (even if the writing of that book delayed this one by a few years).

As regards the genesis of this book, my friends and colleagues Julian Bradfield, Mike Fourman, Peter Sewell, Alex Simpson and Perdita Stevens responded positively to the initial concept and helped me to believe that the idea was worth pursuing. The University of Edinburgh granted me a sabbatical in the academic year 2009–10 which enabled me to make a serious start on the project, and another in 2018–19 which helped me see the initial draft to completion.

Preliminary feedback on early versions of some chapters was given by my colleagues Alex Simpson and Martín Escardó, leading to some substantial changes. Bahareh Afshari and Graham Leigh rescued me from a serious misunderstanding in an early draft of Chapter 8. I have also benefited from conversations with Dana Scott, and in particular from his personal reminiscences of some of the logicians who feature in the story. Other valuable feedback and encouragement was provided at various stages by New Anutrakulchai, Dave Berry, Thomas Bradley, Christopher and Kirsten Bradshaw, Vlad Grigoras, Codrin Iftode, Richard Jack, Bruce Kapron, Karoliina Lehtinen, Sam Lindley, Jason Malcolm-Herzmark, Andrew Mott, Grant Passmore, Benjamin Pierce and Peter Williams. Special thanks too to Cat Outram for listening patiently to my hazy ideas regarding the cover artwork, and for the outstanding skill and care she lavished on it.

I am particularly grateful to those friends who took the time to read large parts of the text and provide me with detailed feedback. At an early stage,

these included two younger readers to whom I would like to pay particular tribute. Robin Bradfield astonished me by reading, enjoying and largely understanding the first ten story chapters before reaching the age of 10. And Elizabeth Yallop, at the age of 12, read subsequent drafts of these same chapters and emailed me with a list of thoughtful comments. The enthusiasm of these two young people for the project meant more to me than they could have known at the time.

Much later, in 2020, I benefited from the assistance of Sofia Kamenova, an LFCS summer intern student who worked through the text and made many helpful suggestions. In 2022, I was privileged to receive detailed feedback from David Paton-Williams, himself an experienced writer, who engaged deeply with the story in its entirety and offered a wealth of perceptive comments, many of which led to significant re-workings and enhancements. The resulting version was then read by Alex Simpson, whose enthusiasm and expert feedback helped me to believe I was almost there.

My heartfelt thanks to all my family and friends who have encouraged and supported me during the long gestation of the book, even when they haven't really understood what it was about. My sister Lizzie provided crucial feedback and encouragement over a long period, and became the first person to read to the end of the story. But perhaps predictably, my last and greatest thanks must go to my wife Caroline for her belief in and support for the project at every stage, for helping me to iron out difficulties and obscurities in the text, and above all for her timely and gentle reminders that I really *did* want to write this book, despite all appearances to the contrary.

John Longley
Edinburgh, November 2023

CASTLES IN THE AIR

CHAPTER 1

The noble art of proof

Emily loved numbers. She loved their gleaming colours, their pristine sharpness, their cool, silvery feel as they ran through her fingers. She would sit happily on the old courtyard floor for hours, surrounded by thousand on thousand of them, like hard, polished pebbles of onyx, beryl, topaz, amethyst and lapis lazuli.

Numbers were her friends. Each number, once you got to know it, had a distinctive character or personality of its own.

Take 24, for instance: ruby-red, translucent, and a *factorial* number ($1\times2\times3\times4$), its crystalline form hinting at the array of smaller numbers that divided it exactly (1, 2, 3, 4, 6, 8, 12). Or 25: a blue gem of elegant simplicity, and a perfect square (5×5). The next number, the sandy-brown 26, seemed less distinguished as far as Emily could tell; but then came the saffron 27, a perfect cube ($3\times3\times3$), and the remarkable emerald-green 28, both a *triangular* number ($1+2+3+4+5+6+7$) and a *perfect* number — one that was equal to the sum of its own divisors ($28=1+2+4+7+14$). And so on, and so on, for as long as you cared to look.

There were many games you could play with numbers. Emily knew, of course, how to *multiply* two numbers by spinning them together with a deft flick of the wrist, creating a glistening blur out of which a third, larger number would emerge. More interesting, however, was the opposite game: to take a large number, and to try to break it apart into smaller numbers that would multiply to give the original one. There was no knowing in advance how easy or hard this might be. A number like 1009008 would fall apart at the slightest tap of her hammer into its ten constituents:

$$2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7 \times 11 \times 13 \;=\; 1009008$$

By contrast, the following number, 1009009, had to be struck far more carefully — not hard, but at just the right angle — in order to split it into its two factors, 821 and 1229. Doing this might require some trial and error, but would give a certain satisfaction, like cracking a code — as though by splitting a number you forced it to reveal something of its soul.

Then there were other numbers, like 2, 3, 5, 29 or 1009007, that *couldn't* be split into anything smaller: rather, they themselves were the basic, indivisible building blocks into which all other numbers would split. These — the so-called *prime numbers* — were perhaps the most intriguing customers of all. Primes were generally round and smooth like marbles: of all conceivable colours and hues, but opaque, inscrutable, and hard as diamond. Emily had long ago fallen under their spell, and had collected a big pile of them. She always got a tingle from finding a new prime, particularly a large one. Not that they were especially rare, but they couldn't be found by any simple formula or recipe that she knew. You just had to go hunting.

Emily had often wondered about this as she paced the flagstones that framed the sea of pebbles. The way the primes were scattered among the other numbers seemed random and chaotic, like grains shaken from a pepper-pot. Looking for primes was like waiting for buses: there could be a gap of 100 or more with no primes at all, then five might come in a rush. How could the perfect, pristine, eternal realm of numbers tolerate such anarchy, such apparent disorder?

In fact, there was only one trend in the sequence of primes that Emily had ever noticed. The higher you looked, the rarer the primes seemed to be. That was to be expected, of course. To be prime, a number had to avoid being divisible by any of the numbers below it — so the larger the number, the more hurdles it would have to clear. A number up in the millions would need to pass far more tests than one in the hundreds in order to be prime. So it wasn't surprising that primes were scarcer among the millions than among the hundreds.

But this simple observation raised a curious question. Maybe if you looked at *really* big numbers, the primes would become rarer and rarer until… well, could it be that eventually they would dry up completely, and there would be *no primes at all* beyond a certain point? The very idea seemed slightly bizarre, and not especially likely. But was it *possible*, Emily asked herself, that that would eventually happen?

* * * * *

"No!" said a voice behind her, startling her from her reflections.

Emily turned round in alarm.

A man in a flowing grey robe was standing in the cloisters beneath the stone buildings that enclosed the courtyard. He had a grand and noble appearance, with a long beard and highly domed forehead, like a marble sculpture in a museum. He was holding a long metal ruler and a large pair of compasses. He looked older than anyone Emily had ever seen.

"As you suspected, the primes continue for ever. There is no end to them."

Emily stared at him. Who was this curious old man? And how could he possibly *know* what he was telling her? He did, admittedly, seem so aged and venerable that Emily could almost imagine he really had tested every number in the universe…

The man seemed to read her thoughts. “I may be old,” he said, “but of course, I am only *finitely* old: about 2,300 years old, in fact. And even you, child, cannot suppose that in that time I have contrived to examine the entire infinity of numbers one by one. No: such truths as this cannot be reached by the drudgery of mere number-crunching. They may be attained only by the exalted, sublime and honourable discipline of Proof: that rare and subtle exercise of Logic and Reason.”

Emily felt somewhat affronted by his overbearing manner, but she was both too frightened and too curious to interrupt.

His tone seemed to soften a little. “It is clear,” he continued, “that you have a real love of numbers and a genuine thirst for knowledge. And that, indeed, is already a large part of what it takes to make a real mathematician. But to uncover the deeper truths — the hidden secrets of the Castle — you will need to go beyond mere calculation.”

He gestured mysteriously to the surrounding buildings, rising by three or four storeys to a zigzag of sharp gables with high latticed windows.

Emily finally plucked up her courage. “Who are you?” she asked.

The man smiled. “My name is Euclid,” he said, sitting down on the steps at the edge of the pebbled floor. “I am, or rather was, a Greek mathematician — an *ancient* Greek, as you would now say. I spent most of my time in another part of the Castle, as I was mainly interested in geometry: in lines, triangles, polygons, circles and their properties.”

He nodded towards his ruler and compasses. “When I was young, I would spend many happy hours drawing triangles and circles in the sand, much as you now love playing with numbers. But then I learned the noble art of Proof, which gave me the key to far deeper truths — truths that apply to *all* triangles, to *all* circles, not just the particular ones I happened to have drawn. Truths, moreover, that can be established beyond a shadow of doubt, being derived from indisputable first principles by the unassailable laws of logic. But I also spent some time studying numbers — once again, not just particular numbers, but questions relating to the whole infinity of numbers. Here, too, I discovered Proofs of some interesting facts — including, as it so happens, the answer to your question about the primes.”

Something clicked in Emily’s mind. “Oh yes,” she said, “*I* know what you’re talking about! Things that are true for all numbers — yes, *algebra*, I’ve heard of that. Using letters to stand for any number you like: $x^2 - 1$ is $x + 1$ times $x - 1$, that kind of thing. Yes! I don’t know much about it myself, but my older sister Anna, she’s very good at it.”

Euclid, however, seemed unimpressed by this display of scholarship: it appeared that algebra was not what he had had in mind.

"It is true," he said patiently, "that algebra is a very useful tool — and one which, I regret, we did not have in my day. And it is also true that deductions in algebra are indeed examples of Proofs, albeit Proofs of a rather humble kind. But" — he resumed his elevated manner — "algebra by itself, or rather what your sister knows as algebra, is powerless in the face of the deeper Mysteries. You cannot hope, for example, to settle your question about the primes in that way. Those who know nothing beyond algebra can have little conception of the true character of mathematics: the logical cut-and-thrust of argument, claim, premise, conjecture, assertion and rebuttal; the majestic architecture of theorem rising upon theorem in a mighty, unshakeable edifice; the brilliance of an ingenious strategy unfolding like a master battle plan; the gradual tightening of a net of inescapable logic — and above all, that moment of glory when the truth dawns, the crux is resolved, the jigsaw falls into place, and the Proof is complete."

Emily's curiosity was now thoroughly engaged, although all these allusions to some secret wisdom beyond her comprehension were starting to grate with her. She tried to voice her thoughts.

"Well, prime numbers certainly fascinate me," she began. "So if there really is some way to know for sure that they go on for ever, then I'd love to know about it. But — well, forgive me, but this Proof business you're describing all sounds terribly difficult. Isn't it — you know, just a bit too *lofty* for us ordinary mortals?"

Euclid laughed heartily, and his whole manner seemed to lighten considerably.

"I fear I may have misled you," he said. "I was thinking mainly of the process of *discovering* proofs: of proving new things that no one has ever proved before. There is a world of difference between that, and the matter of simply *understanding* a proof that someone else has already found. It is the difference between discovering some buried treasure for yourself amidst a vast landscape, and finding it once someone has given you a map showing the exact spot. The latter might still involve some effort, and perhaps a lengthy trek, but no very exceptional skill or detective work. As for the proof I mentioned: you will, I think, find it well within your grasp. Indeed, you may be surprised how near to hand it lies."

He paused, and his eyes assumed a faraway look. "But finding something new in mathematics — I mean something *interesting* and new, something really worth knowing — now there is a challenge to tax even the greatest of intellects. Especially in an area like number theory, where so much of the landscape has been trodden and re-trodden for hundreds of years already. That, arguably, is an occupation for a few minds only. Though I daresay," he added with a smile, "your chances of success would be as good as any."

Emily's eyes lit up. Much of what Euclid was saying was now starting to make sense to her: in fact, it was sounding strangely familiar.

"So…," she ventured, "is it a bit like the difference between cracking some eight-digit number for myself using my hammer here — which is sometimes very difficult — and simply checking, if someone else tells me what the factors are, that they do indeed multiply together to give the number in question — which is easy and boring by comparison?"

Euclid thought for a moment.

"Yes," he said, "I suppose it *is* like that. Only on a much higher level. Factoring an eight-digit number is mostly a matter of brute perseverance, I think you'd agree. Whereas finding proofs requires imagination, creativity, a good instinct, a dash of courage, and perhaps a measure of good luck. It is not usually practicable to find interesting proofs just by trial and error. But fundamentally, you are right: the difference is of that kind."

"And… this proof you were talking about, that there are infinitely many primes — is that really something I would be able to understand for myself?"

"I should be delighted to show it to you," said Euclid, casting a glance towards the shadowy cloisters. But first, we will need these, if we intend to do any serious exploring."

He reached into the folds of his robe, and drew out a bunch of strange metal objects of different sizes, each twisted into some curious form.

"You will get quite a long way with these," he said smiling.

"What are they?"

"The keys to the kingdom, child! They are used for opening different kinds of doors in the Castle. Each one corresponds, if you like, to some principle of reasoning, some tactic that can be used in proofs. This one here, for instance," — he held up a piece of metal shaped into what looked like a small silver ladder — "is called the Induction Principle. It's a useful way of showing that something is true for *all* numbers. Rather like climbing a ladder: if we can get ourselves onto the bottom step, and can get from any step to the one above it, then we are able to climb as high as we like. You'll get the idea soon enough."

He picked out two of the other keys. "This one here is the Principle of Cases, which lets us separate a problem into two sub-problems: for instance, we might prove something first when $x \leq y$ and then when $x > y$. This one is called the Antisymmetry Principle: if we can prove both $x \leq y$ and $y \leq x$, then x and y must be equal. And there are others too, whose use I'm sure you'll discover for yourself easily enough."

This lecture had been delivered at such a breathless speed, and seemed to leave so much unexplained, that Emily felt far from sure she would be able to master the use of the keys as easily as Euclid seemed to anticipate.

"Don't worry," said Euclid. "It becomes much easier once you see them in action. But today we are going to use this key, one of my favourites."

He selected a black iron key like an elaborately wrought cross. "The Principle of Contradiction!" he said. "You shall see what it does soon enough. Come with me."