APPENDIX A

Chapter commentaries

The following commentaries provide some intellectual and historical context for the ideas introduced in the story, developing the concepts a little further and indicating how they fit into the big picture.

Fuller versions of (some of) these commentaries are freely available from the book's website. Many points that are only lightly touched on or hinted at here are explored in more depth in these longer commentaries. The website also offers a number of supplements dedicated to specific topics — some of these are referred to below.

Chapter 1: The noble art of proof

We've begun our story with a proof of a classic result from number theory: There are infinitely many prime numbers. This was essentially the proof given by the Greek mathematician Euclid of Alexandria (c. 325-265 BCE) in his monumental treatise, the *Elements*. The proof itself might well have been the work of some earlier mathematician, but it's traditionally known as 'Euclid's proof', and in the story I've taken the liberty of ascribing it to him.

This book isn't really about number theory. It's really about a subject called *mathematical logic*, and a bunch of questions that lie on the border between mathematics and philosophy. But in order to approach these questions — to set the scene — it's as well to start by looking at some actual examples of mathematics. And number theory serves this purpose very well.

What is so special about mathematics? What is it that distinguishes it from other subjects? Various answers to this question might be given, but certainly one distinctive feature of mathematics is the notion of rigorous, logical *proof*.[1]

What *is* a mathematical proof? Actually, a completely precise answer to this question isn't so easy to give, and that's one of the main things this book will be about. But the basic idea is clear enough: a proof is just a logically watertight argument that shows, beyond all possible doubt, why some mathematical statement is true. Statements that have been proved in this way

1 The focus of this book is mainly on what is often called *pure mathematics* — broadly speaking, mathematics studied for its own sake rather than with a view to applications — though applications will often be touched on in our discussions.

are known as *theorems* — they are represented in our story by the medallions which our heroines will spend much of their time searching for. Euclid's theorem on the infinitude of the primes is a classic example.

So a proof in mathematics is something quite different from a mere accumulation of data that might be seen as 'evidence' for some claim. For example, there's a famous conjecture made by Christian Goldbach in 1742 which says that every even number greater than 2 is the sum of two prime numbers (e.g. $42 = 19 + 23$). This has been checked by computer for all even numbers up to 4,000,000,000,000,000,000, but that doesn't prove that it holds for *all* even numbers — and indeed, the challenge of finding a rigorous proof of Goldbach's conjecture still remains open. There are even some mathematical statements that appear to hold for all numbers up to 10^{300}, and yet are known to fail for some larger numbers. So mathematicians as a breed tend to be highly mistrustful of mere cumulative evidence.

Mathematics as a science of measurement and calculation had been extensively developed in pre-Greek times by the Babylonians, the Chinese, the Indians and others. But it was seemingly the Greek philosopher Thales of Miletus (c. 624-545 BCE) who was the first to hit on the principle of mathematical proof: the idea that truths about *all* numbers, or *all* triangles, or whatever, could be securely established by means of a sequence of logical steps. In effect, what this amounted to was a completely new *way of knowing things*: a milestone in the history of knowledge whose importance is hard to overestimate. By Euclid's time, this idea had developed into a methodology of deducing a whole mathematical theory from an explicitly stated list of *axioms* — initial assumptions that were seen as so self-evident that they did not themselves require justification.

One cannot but be impressed by how well the mathematics of antiquity has lasted. Ancient Greek ideas on the composition of matter or the movements of the planets were discarded long ago, but their mathematics has endured (even if we'd now see their discoveries in geometry in a somewhat different light). Why is this? One obvious answer suggests itself: knowledge acquired by mathematical proof is by its nature *certain*, and therefore indestructible.

So the idea of mathematical proof is closely linked to the idea that mathematics offers some sort of 'ideal certainty' of a kind not available in other disciplines. This is an enormously potent idea: it has gripped many minds down the centuries, and it is surely a part of what draws many people to mathematics.

How certain is 'certain'? Can we really say, as some thinkers have been eager to claim, that mathematics offers nothing short of *absolute* certainty?

One might think of ways in which this bold claim might need at least some qualification. Some of these will be explored in the course of this book, and a few others are discussed in the extended commentary. Nonetheless, even allowing for these caveats, there still remains a basic sense that mathematics

— or at least some of it — is *in some way* peculiarly reliable and certain. In number theory, for instance, we find a significant body of crystal-clear knowledge that is not, in fact, disputed or doubted by anyone at all. There may be some sophisticated people around who like to tell us that 'mathematical certainty' is an obsolete and discredited notion, but in practice, even these people find a proof in number theory completely compelling when they see one. For example, it was proved by Leonhard Euler around 1760 that there are no positive Whole numbers x, y, z such that $x^3 + y^3 = z^3$. And no one who understands the proof thinks there's the slightest point in looking for any such numbers — because we all seem to agree, at bottom, that a mathematical theorem is something it would be *irrational to deny*.

Although this notion of mathematical certainty may on one level seem quite clear and familiar, it gives rise to some perplexing philosophical puzzles. How is it, exactly, that our minds are able to acquire this 'certain' knowledge, with little or no reference to the outside world? What goes on inside our heads — biologically speaking — when we 'see' that some mathematical statement must be true? And what is it about this process that makes the resulting knowledge so distinctively 'certain'?

Such questions may seem particularly puzzling when we consider mathematical statements that involve the notion of *infinity* in some way. How is it that our (presumably) finite minds are able to acquire knowledge of the infinite? What process within our brains could possibly deliver reliable knowledge about infinity?

Could a digital computer be programmed to 'see' mathematical truths with complete certainty in the way that we can? If so, could the computer likewise be programmed to 'see' things that *we* can see to be false?

These are far from easy questions, and as we shall see later in this book, different thinkers have responded to them in radically different ways. But lurking in the background to these puzzles, there is perhaps an even more fundamental conundrum. What *is* mathematical truth? We evidently possess mathematical knowledge of some kind, but what exactly is it knowledge *of*? What does it *mean* to say there are infinitely many primes? To put the question very crudely: *Where* are there infinitely many primes?

Again, as we shall see, there is a whole spectrum of possible ways of either addressing or dismissing such questions. But here we'll make a start by touching on one of the oldest of these: the view known as *platonism*, so named after the philosopher Plato who flourished in Athens around 360 BCE.

The essence of mathematical platonism (or Platonism: take your pick) is simply the idea that there is some kind of objective truth or reality, independent of ourselves, to which mathematical statements refer. Just as there is (we commonly suppose) a physical world outside ourselves to which we refer when we speak of trees and elephants, so there is some realm of mathematical reality outside ourselves to which we refer when we speak of prime numbers or perfect circles. This platonic realm of reality is typically held to be abstract, non-physical, timeless and immutable.[2]

The idea of platonism sharply divides people. There are many top-flying mathematicians today to whom the platonist view seems perfectly natural and reasonable, and others to whom the very idea of such a 'mathematical world' seems ridiculous, fanciful, unbelievable. So what exactly is going on here? Do the platonists know something the anti-platonists don't, or *vice versa*? Or are they somehow talking at cross-purposes and misconstruing each other's positions? Is the entire debate ultimately a meaningless one arising from some confusion of language? Are there different degrees or levels of platonism that need to be distinguished? If we reject platonism, what other view of mathematical truth might we adopt in its place?

Again, these are not easy questions. We shall try to unpack them as we proceed, and consider arguments from a range of perspectives. But in the meantime, we can note one thing that platonists and anti-platonists alike largely agree on: namely, that doing mathematics certainly *feels like* exploring a world that's already out there — a world we 'discover' rather than one we 'invent'. Whether that's really what's going on is of course another matter. But the deeper and more fundamental the mathematics, the more compelling this impression of discovery seems to be.

In the story, I've tried to capture something of this sense of exploration and discovery using the metaphor of a huge fantasy castle. It might seem that this metaphor leans towards a platonistic view of mathematical reality, but that is not really the intention. The purpose is simply to set the scene by portraying something of what mathematics feels like from the inside. What the real nature of mathematics might be is something I invite you to reflect on as the story unfolds.

Chapter 2: Beads and bracelets

In Chapter 1, we introduced the notion of theorem and proof by looking at just a single example. In this chapter, we shall look at another example, that of *Fermat's little theorem* (not to be confused with his celebrated *last theorem* — an incomparably harder result which was finally established by Andrew Wiles in 1994). But we shall also start to glimpse how theorems and other ingredients can combine to form something much bigger.

The 'little theorem' was stated by Fermat in a letter to a friend in 1640, though the first published proof was given by Euler in 1736. Many proofs are

2 The term 'platonism' is popular among mathematicians, while philosophers often use the broad term 'realism' for any view that sees mathematics as dealing with objective, mind-independent truth. Some scholars reserve 'platonism' for a position more strictly in line with Plato's specific ideas.